Lobatto and Radau positive quadrature formulas for linear combinations of Jacobi polynomials

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Abstract

For a given $\theta \in (-1,1)$, we find out all parameters $\alpha, \beta \in \{0,1\}$ such that, there exists a linear combination of Jacobi polynomials $J_{n+1}^{(\alpha,\beta)}(x) - CJ_n^{(\alpha,\beta)}(x)$ which generates a Lobatto (Radau) positive quadrature formula of degree of exactness $2n+2$ $(2n+1)$ and contains the point θ as a node. These positive quadratures are very useful in studying problems in one-sided polynomial L_1 approximation.

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1 Introduction

Let σ be a positive measure on $[-1,1]$ such that the support of $d\sigma$ contains an infinite set of points. Fix points $-1 < x_{n,1} < x_{n,2} < \ldots < x_{n,n} < 1$ and consider a quadrature formula of the form

$$\int_{-1}^{1} f(x)d\sigma(x) = \sum_{j=1}^{n} \lambda_{n,j} f(x_{n,j}) + R_n(f), \tag{1}$$

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with $\lambda_{n,j} > 0$. For $0 \le m \le n$, this formula is called a *positive* $(2n-1-m, n, d\sigma)$ *quadrature formula* [5], if $R_n(P) = 0$, for all $P \in \mathbb{P}_{2n-1-m}$ (as usual, $\mathbb{P}_n$ denotes the set of all polynomials of degree at most n). The number $2n-1-m$ is the degree of exactness.

We say that a polynomial $P_n \in \mathbb{P}_n$ generates a positive $(2n-1-m, n, d\sigma)$ quadrature formula, if P_n has n simple zeros $x_{n,1} < x_{n,2} < \ldots < x_{n,n}$ in $(-1,1)$ and the interpolatory quadrature formula based on the nodes $x_{n,j}$ is a positive $(2n-1-m, n, d\sigma)$ quadrature formula.

We say that a polynomial Q_{n+1} generates a Radau positive $(2n+1, n+1, dx)$ quadrature formula, if Q_{n+1} has $n+1$ simple zeros $y_{n+1,1} < y_{n+1,2} < \ldots < y_{n+1,n+1}$ in $(-1,1)$ and

$$\int_{-1}^{1} P(x)dx = \mu_{n+1,0}P(-1) + \sum_{k=1}^{n+1} \mu_{n+1,k}P(y_{n+1,k}), \quad \text{for all} \quad P \in \mathbb{P}_{2n+1}, \tag{2}$$

where the weights $\mu_{n+1,k}$ $(0 \le k \le n+1)$ are positive. For the node 1, we have a similar definition. In this case

$$\int_{-1}^{1} P(x)dx = \sum_{k=1}^{n+1} \mu_{n+1,k}P(y_{n+1,k}) + \mu_{n+1,n+1}P(1), \quad \text{for all} \quad P \in \mathbb{P}_{2n+1}, \tag{3}$$

We say that a polynomial Q_{n+1} generates a Lobatto positive $(2n+2, n+1, dx)$ quadrature formula, if Q_{n+1} has $n+1$ simple zeros $y_{n+1,1} < y_{n+1,2} < \ldots < y_{n+1,n+1}$ in $(-1,1)$ and

$$\int_{-1}^{1} P(x)dx = \mu_{n+1,0}P(-1) + \sum_{k=1}^{n+1} \mu_{n,k}P(y_{n,k}) + \mu_{n,n+2}P(1), \tag{4}$$

for all $P \in \mathbb{P}_{2n+2}$, where the weights $\mu_{n+1,k}$ $(0 \le k \le n+2)$ are positive.

We remark that, in studying some problems related with best one-sided approximation (see [1]), there appear positive quadrature formulas related with linear combinations of Jacobi polynomials of the form

$$T_{n+1}^{(\alpha,\beta)}(x) = J_{n+1}^{(\alpha,\beta)}(x) - C\, J_n^{(\alpha,\beta)}(x), \tag{5}$$

where $J_k^{(\alpha,\beta)}(x)$ is the monic Jacobi polynomial of degree k. In particular, when we want to characterize the polynomials of the best one-sided L_1-approximation to Heaviside functions (in this case $\theta \in (-1,1)$ is the parameter defining the Heaviside function), the following problem arises: Does the polynomial $T_{n+1}^{(\alpha,\beta)}$ defined by (5) generate a Lobatto (Radau) type positive quadrature formula? As we will show here the analysis is not simple, because for $n \ge 1$ and $\alpha, \beta \in \{0, 1\}$ fixed, we can have such a positive quadrature only in some specific subintervals of $(-1,1)$, while we need to cover all the open interval $(-1,1)$. Therefore, some changes of the parameters α and β are necessary, and we need to know when and how the changes should be done. These facts were the motivation for writing this paper.

Positive quadrature formulas associated to linear combination of orthogonal polynomials have been studied by different authors (for instance, see [6] and [7] and the references therein). In particular Theorem 4.1 of [7] provides sufficient (and complicated) conditions to obtain Lobatto and Radau type positive quadrature formula associated to Jacobi weights, with $\alpha, \beta \in \{0, 1\}$. In [7] the author deals with general orthogonal polynomials. Here we need a more careful analysis. That is the reason why we pay attention to Theorem 2.1 of [7], instead of Theorem 4.1. The use of special properties of Jacobi polynomials helps us to obtain more specific results. Moreover, we will present the result it terms which depends on the parameters α and β defining the Jacobi polynomials.

The paper is organized as follows. In Section 2 we collect several known facts related with Jacobi polynomials which will be needed. In Section 3, for $n \in \mathbb{N}_0$ and $\alpha, \beta \in \{0, 1\}$, we first find all of real C for which the polynomial $T_{n+1}^{(\alpha,\beta)}$ given in (5) generates a positive $(2n, n+1, (1-x)^\alpha(1+x)^\beta dx)$ quadrature formula. Some of these C generates a Lobatto (Radau) type positive quadrature formula, others not. For the case when $\alpha + \beta \neq 0$, we find out all C for which there is a Lobatto (Radau) formula. In the last section we consider a more complicated problem. For $\theta \in (-1, 1)$ and $n \in \mathbb{N}_0$ fixed, we want to find out all Lobatto (Radau) positive quadrature, with degree of exactness $2n + 2$ $(2n + 1)$, which contains θ as a node. Of course, the problem has interest only when θ itself is not a zero of a Jacobi polynomial of degree n (with $\alpha, \beta \in \{0, 1\}$). That is the reason why we define

$$Q_n^{(\alpha,\beta)}(x) = \frac{J_{n+1}^{(\alpha,\beta)}(x)}{J_n^{(\alpha,\beta)}(x)} \qquad (J_n^{(\alpha,\beta)}(x) \neq 0), \tag{6}$$

and consider the linear combination

$$R_{n+1,\theta}^{(\alpha,\beta)}(x) = J_{n+1}^{(\alpha,\beta)}(x) - Q_n^{(\alpha,\beta)}(\theta) J_n^{(\alpha,\beta)}(x), \tag{7}$$

where $\theta \in (-1, 1)$. The question is: for which selections of α and β ($\alpha, \beta \in \{0, 1\}$) the polynomial $R_{n+1,\theta}^{(\alpha,\beta)}(x)$ generates a Lobatto (Radau) type positive quadrature? This is just the problem which appears in studying best one-sided polynomial L_1-approximation for the Heaviside functions. We will solve it. For the solution we use some recent results related with the interlacing properties of zeros of the Jacobi orthogonal polynomials given in [3].

2 Some facts related to Jacobi polynomials

For $\alpha > -1$, $\beta > -1$ and $n \in \mathbb{N}$, the Jacobi polynomial $P_n^{(\alpha,\beta)}$ is the unique polynomial of degree n which satisfies

$$\int_{-1}^{1} Q_{n-1}(x) P_n^{(\alpha,\beta)}(x)\,(1-x)^\alpha (1+x)^\beta\,dx \;=\; 0, \quad \text{for all}\, Q_{n-1} \in \mathbb{P}_{n-1}, \tag{8}$$

and

$$P_n^{(\alpha,\beta)}(1) \;=\; \binom{n+\alpha}{n}. \tag{9}$$

It is known that (see [9], p. 59)

$$P_n^{(\alpha,\beta)}(x) \;=\; (-1)^n\, P_n^{(\beta,\alpha)}(-x) \quad \text{and} \quad P_n^{(\alpha,\beta)}(-1) \;=\; (-1)^n \binom{n+\beta}{n}. \tag{10}$$

The coefficient of x^n in $P_n^{(\alpha,\beta)}$ (see [9], p. 63) is given by

$$l_n^{(\alpha,\beta)} = \frac{1}{2^n}\binom{2n+\alpha+\beta}{n}. \tag{11}$$

The monic Jacobi polynomial will be denoted by $J_n^{(\alpha,\beta)}(x)$. Since

$$J_n^{(\alpha,\beta)}(x) = \frac{1}{l_n^{(\alpha,\beta)}}\, P_n^{(\alpha,\beta)}(x), \tag{12}$$

one has

$$J_n^{(\alpha,\beta)}(1) = \frac{2^n\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}\,\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \tag{13}$$

and

$$J_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{2^n\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}\,\frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)} \tag{14}$$

The Jacobi polynomials satisfy a recurrence relation (see [9], p. 71):

$$P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = [(\alpha+\beta+2)x+\alpha-\beta]/2$$

and, for $n \geq 2$,

$$\begin{gathered} 2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{(\alpha,\beta)}(x) \\ = (2n+\alpha+\beta-1)[(2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2]P_{n-1}^{(\alpha,\beta)}(x) \\ -2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(x). \end{gathered} \tag{15}$$

For the monic Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$, with $n \geq 2$, one has (see [2], p. 153)

$$J_n^{(\alpha,\beta)}(x) = (x - c_n^{(\alpha,\beta)})J_{n-1}^{(\alpha,\beta)}(x) - \lambda_n^{(\alpha,\beta)} J_{n-2}^{(\alpha,\beta)}(x), \tag{16}$$

where

$$c_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta-2)(2n+\alpha+\beta)} \tag{17}$$

and

$$\lambda_n^{(\alpha,\beta)} = \frac{4(n-1)(n+\alpha-1)(n+\beta-1)(n+\alpha+\beta-1)}{(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-1)(2n+\alpha+\beta-3)}. \tag{18}$$

When $n = 1$ we should take $c_1^{(\alpha,\beta)} = (\beta-\alpha)/(\alpha+\beta+2)$.

From (11), it follows that $l_n^{(\alpha,\beta)} = l_n^{(\beta,\alpha)}$. Hence, from (10) and (12) one has

$$Q_n^{(\alpha,\beta)}(-x) = \frac{l_n^{(\alpha,\beta)}}{l_{n+1}^{(\alpha,\beta)}}\,\frac{P_{n+1}^{(\alpha,\beta)}(-x)}{P_n^{(\alpha,\beta)}(-x)} \;=\; -\,\frac{l_n^{(\alpha,\beta)}}{l_{n+1}^{(\alpha,\beta)}}\,\frac{P_{n+1}^{(\beta,\alpha)}(x)}{P_n^{(\beta,\alpha)}(x)} \;=\; -\,Q_n^{(\beta,\alpha)}(x). \tag{19}$$

Moreover, from (13) and (14), we obtain

$$Q_n^{(\alpha,\beta)}(1) = \frac{2(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)} \;>\; 0 \tag{20}$$

and

$$Q_n^{(\alpha,\beta)}(-1) = -Q_n^{(\beta,\alpha)}(1) \;=\; -\frac{2(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)} \;<\; 0. \tag{21}$$

A direct calculation shows that

$$-1 - Q_n^{(\alpha,\beta)}(-1) - \frac{\lambda_{n+1}^{(\alpha,\beta)}}{Q_{n-1}^{(\alpha,\beta)}(-1)} = c_{n+1}^{(\alpha,\beta)} = 1 - Q_n^{(\alpha,\beta)}(1) - \frac{\lambda_{n+1}^{(\alpha,\beta)}}{Q_{n-1}^{(\alpha,\beta)}(1)} \tag{22}$$

It is known that the zeros of the Jacobi polynomials are real, distinct and are located on the interior of the interval $[-1,1]$ (see [9], p. 44). For $n \in \mathbb{N}$, the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ will be denoted by $x_{n,k}^{(\alpha,\beta)}$ $(1 \le k \le n)$, with

$$-1 < x_{n,1}^{(\alpha,\beta)} < \cdots < x_{n,n}^{(\alpha,\beta)} < 1.$$

The zeros have the interlacing property (see [9], p. 46 or [2], p. 28):

$$x_{n+1,1}^{(\alpha,\beta)} < x_{n,1}^{(\alpha,\beta)} < x_{n+1,2}^{(\alpha,\beta)} < x_{n,2}^{(\alpha,\beta)} < \ldots < x_{n,n}^{(\alpha,\beta)} < x_{n+1,n+1}^{(\alpha,\beta)}. \tag{23}$$

In what follows, for simplicity we use general parameters $\alpha, \beta > -1$, but we need only the cases $\alpha, \beta \in \{0,1\}$.

3 Quadrature formulas

Proposition 1. *For $\alpha, \beta > -1$, $n \in \mathbb{N}$, the polynomial $T_{n+1}^{(\alpha,\beta)}(x)$ given in* (5) *generates a positive $(2n, n+1, (1-x)^\alpha(1+x)^\beta dx)$ quadrature formula if and only if*

$$Q_n^{(\alpha,\beta)}(-1) \;<\; C \;<\; Q_n^{(\alpha,\beta)}(1). \tag{24}$$

Proof. It follows from e) in Theorem 2.1 of [7] (with $m = 1$) that $T_{n+1}^{(\alpha,\beta)}(x)$ generates a positive $(2n, n+1, (1-x)^\alpha(1+x)^\beta dx)$ quadrature formula if and only if the following conditions hold

i) there exists a constant K, with $|K| < 1$, such that

$$T_{n+1}^{(\alpha,\beta)}(x) = (x+K)J_n^{(\alpha,\beta)}(x) - \lambda_{n+1}^{(\alpha,\beta)}\, J_{n-1}^{(\alpha,\beta)}(x),$$

where $\lambda_{n+1}^{(\alpha,\beta)}$ is given by (18), and

ii) $\operatorname{sgn}\big(T_{n+1}^{(\alpha,\beta)}(\pm 1)\big) = (\pm 1)^{n+1}$.

Notice that, if $J_n^{(\alpha,\beta)}(x) \neq 0$, then

$$T_{n+1}^{(\alpha,\beta)}(x) = J_{n+1}^{(\alpha,\beta)}(x) - C\, J_n^{(\alpha,\beta)}(x) = J_n^{(\alpha,\beta)}(x)\big(Q_n^{(\alpha,\beta)}(x) - C\big).$$

So,

$$T_{n+1}^{(\alpha,\beta)}(1) = J_n^{(\alpha,\beta)}(1)\big(Q_n^{(\alpha,\beta)}(1) - C\big)$$

and

$$T_{n+1}^{(\alpha,\beta)}(-1) = J_n^{(\alpha,\beta)}(-1)\big(Q_n^{(\alpha,\beta)}(-1) - C\big).$$

From (14) we conclude that the condition ii) is just equivalent to (24). So, we only need to prove that (24) implies i).

Indeed, from (16), we can write

$$T_{n+1}^{(\alpha,\beta)}(x) = \big(x - c_{n+1}^{(\alpha,\beta)} - C\big) J_n^{(\alpha,\beta)}(x) - \lambda_{n+1}^{(\alpha,\beta)} J_{n-1}^{(\alpha,\beta)}(x).$$

Thus, it is sufficient to prove that $\mid c_{n+1}^{(\alpha,\beta)} + C \mid < 1$. But, from (22) we obtain

$$-1 < -1 - Q_n^{(\alpha,\beta)}(-1) - \frac{\lambda_{n+1}^{(\alpha,\beta)}}{Q_{n-1}^{(\alpha,\beta)}(-1)} + Q_n^{(\alpha,\beta)}(-1) = c_{n+1}^{(\alpha,\beta)} + Q_n^{(\alpha,\beta)}(-1)$$

$$< c_{n+1}^{(\alpha,\beta)} + C < c_{n+1}^{(\alpha,\beta)} + Q_n^{(\alpha,\beta)}(1) = 1 - Q_n^{(\alpha,\beta)}(1) - \frac{\lambda_{n+1}^{(\alpha,\beta)}}{Q_{n-1}^{(\alpha,\beta)}(1)} + Q_n^{(\alpha,\beta)}(1) < 1.$$

$\square$

For our next result we need the following Radau quadrature (see [4], p. 169).

Theorem 1. *For each $m \in \mathbb{N}$ and every polynomial $P \in \mathbb{P}_{2m}$, one has*

$$\int_{-1}^{1} P(x)dx = A_m\, P(-1) + \sum_{k=1}^{m} R_{m,k} P\big(x_{m,k}^{(0,1)}\big), \tag{25}$$

and

$$\int_{-1}^{1} P(x)dx = B_m\, P(1) + \sum_{k=1}^{m} S_{m,k} P\big(x_{m,k}^{(1,0)}\big), \tag{26}$$

where, for $1 \leq k \leq m$, $R_{m,k}$ and $S_{m,k}$ are positives and

$$A_m = B_m = \frac{2}{(m+1)^2}.$$

Theorem 2. *Fix $n \in \mathbb{N}$.*

i) *The polynomial $T_{n+1}^{(0,1)}(x)$ generates a positive $(2n, n+1, (1+x)dx)$ quadrature formula and a Radau positive $(2n+1, n+1, dx)$ quadrature formula as in (2) if and only if*

$$Q_n^{(0,1)}(-1) < -\frac{n+1}{2n+3} \; < \; C \; < \; Q_n^{(0,1)}(1) = \frac{n+2}{2n+3}. \tag{27}$$

ii) *The polynomial* $T_{n+1}^{(1,0)}(x)$ *generates a positive* $(2n, n+1, (1-x)dx)$ *quadrature formula and a Radau positive* $(2n+1, n+1, dx)$ *quadrature formula as in* (3) *if and only if*

$$Q_n^{(1,0)}(-1) = -\frac{n+2}{2n+3} \;<\; C \;<\; \frac{n+1}{2n+3} < Q_n^{(1,0)}(1). \tag{28}$$

Proof. We will prove i). The proof of ii) is similar.

i) ($\Rightarrow$) From Proposition 1 and (21) we know that

$$-\frac{(n+2)^2}{(2n+3)(n+1)} = Q_n^{(0,1)}(-1) \;< C \;< Q_n^{(0,1)}(1) = \frac{n+2}{2n+3}. \tag{29}$$

Moreover, it follows from (2) that

$$\mu_{n+1,0} \;=\; \frac{1}{T_{n+1}^{(0,1)}(-1)} \int_{-1}^{1} T_{n+1}^{(0,1)}(x)dx.$$

Taking into account that (see Theorem 1), for each $m \in \mathbb{N}$,

$$\int_{-1}^{1} J_m^{(0,1)}(x)dx = \frac{2}{(m+1)^2}\, J_m^{(0,1)}(-1),$$

one has

$$\int_{-1}^{1} T_{n+1}^{(0,1)}(x)dx \;=\; \frac{2}{(n+2)^2}\, J_{n+1}^{(0,1)}(-1) - C\frac{2}{(n+1)^2}\, J_n^{(0,1)}(-1)$$

$$= \frac{2J_n^{(0,1)}(-1)}{(n+1)^2}\left(\frac{(n+1)^2}{(n+2)^2}\, Q_n^{(0,1)}(-1) - C\right) = \frac{2J_n^{(0,1)}(-1)}{(n+1)^2}\left(-\frac{n+1}{2n+3} - C\right).$$

On the other hand

$$T_{n+1}^{(0,1)}(-1) \;=\; J_{n+1}^{(0,1)}(-1) - CJ_n^{(0,1)}(-1) = \; J_n^{(0,1)}(-1)(Q_n^{(0,1)}(-1) - C).$$

Therefore

$$\mu_{n+1,0} \;=\; \frac{1}{(n+1)^2(Q_n^{(0,1)}(-1) - C)}\left(-\frac{n+1}{2n+3} - C\right).$$

Since $Q_n^{(0,1)}(-1) - C < 0$ and $\mu_{n+1,0} > 0$, we conclude that

$$C > -\frac{n+1}{2n+3}.$$

($\Leftarrow$) If (27) holds, then the polynomial $T_{n+1}^{(0,1)}$ generates a positive $(2n, n+1, (1+x)dx)$ quadrature formula for $n > 1$ (Proposition 1). Hence, for every polynomial $P \in \mathbb{P}_{2n}$

$$\int_{-1}^{1} P(x)(1+x)dx = \sum_{k=1}^{n+1} \lambda_{n+1,k} P(y_{n+1,k}), \tag{30}$$

where the points $\{y_{n+1,j}\}_{j=1}^{n+1}$ are the zeros of $T_{n+1}^{(0,1)}$ and

$$\lambda_{n+1,k} = \int_{-1}^{1} \ell_k(x)(1+x)\, dx,$$

where $\ell_k(x)$ are the cardinal Lagrange polynomials at the nodes $\{y_{n+1,j}\}_{j=1}^{n+1}$. Moreover, from (30) we have that, for each $S_{n-1} \in \mathbb{P}_{n-1}$,

$$\int_{-1}^{1} S_{n-1}(x)\, T_{n+1}^{(0,1)}(x)(1+x)dx = 0\,. \tag{31}$$

Note that, every polynomial $P \in \mathbb{P}_{n+1}$ can be written in the form

$$P(x) = P(-1)\frac{T_{n+1}^{(0,1)}(x)}{T_{n+1}^{(0,1)}(-1)} + (1+x)\sum_{k=1}^{n+1} \frac{P(y_{n+1,k})}{1+y_{n+1,k}}\ell_k(x).$$

Hence, from (30), for each $P \in \mathbb{P}_{n+1}$, we have

$$\int_{-1}^{1} P(x)dx = \mu_{n+1,0}P(-1) + \sum_{k=1}^{n+1} \mu_{n+1,j}P(y_{n+1,k}),$$

where, for $1 \le j \le n+1$,

$$\mu_{n+1,j} \;=\; \frac{\lambda_{n+1,j}}{1+y_{n+1,j}} > 0 \quad \text{and} \quad \mu_{n+1,0} \;=\; \frac{1}{T_{n+1}^{(0,1)}(-1)}\int_{-1}^{1} T_{n+1}^{(0,1)}(x)dx > 0,$$

where for the last inequality we have used the computations given in the first part of the proof.

Finally, for every $Q_{2n+1} \in \mathbb{P}_{2n+1}$, there exist polynomials $S_{n-1} \in \mathbb{P}_{n-1}$ and $P_{n+1} \in \mathbb{P}_{n+1}$ such that $Q_{2n+1}(x) = S_{n-1}(x)(1+x)T_{n+1}^{(0,1)}(x) + P_{n+1}(x)$. Taking into account (31), we obtain

$$\begin{aligned}
\int_{-1}^{1} Q_{2n+1}(x)dx &= \int_{-1}^{1} P_{n+1}(x)dx \\
&= \mu_{n+1,0}P_{n+1}(-1) + \sum_{k=1}^{n+1} \mu_{n+1,j}P_{n+1}(y_{n+1,k}) \\
&= \mu_{n+1,0}Q_{2n+1}(-1) + \sum_{k=1}^{n+1} \mu_{n+1,j}Q_{2n+1}(y_{n+1,k}).
\end{aligned}$$

□

For the next result see [4], p. 172.

Theorem 3. *For $m \in \mathbb{N}$ and every polynomial $P \in \mathbb{P}_{2m+1}$, one has*

$$\int_{-1}^{1} P(x)dx = \sum_{k=1}^{m} A_{m,k}P(x_{m,k}^{(1,1)}) + B_mP(-1) + C_mP(1), \tag{32}$$

where, for $1 \le k \le m$, $A_{m,k}$ are positives and

$$B_m = C_m = \frac{2}{(m+1)(m+2)}.$$

Theorem 4. *For $n \in \mathbb{N}$, the polynomial $T_{n+1}^{(1,1)}(x)$ generates a positive $(2n, n+1, (1-x^2)dx)$ quadrature formula and a Lobatto positive $(2n+2, n+1, dx)$ quadrature formula as in (4) if and only if*

$$-\frac{n+1}{2n+3} < C < \frac{n+1}{2n+3}. \tag{33}$$

Proof. ($\Rightarrow$) From Proposition 1 we know that

$$-\frac{n+3}{2n+3} = Q_n^{(1,1)}(-1) < C < Q_n^{(1,1)}(1) = \frac{n+3}{2n+3}. \tag{34}$$

Moreover, it follows from (4) that

$$\mu_{n+1,0} = \frac{1}{T_{n+1}^{(1,1)}(-1)} \int_{-1}^{1} T_{n+1}^{(1,1)}(x)(1-x)dx > 0$$

and

$$\mu_{n+1,n+2} = \frac{1}{T_{n+1}^{(1,1)}(1)} \int_{-1}^{1} T_{n+1}^{(1,1)}(x)(1+x)dx > 0.$$

From Theorem 3, for each $m \in \mathbb{N}$, we have

$$\int_{-1}^{1} J_m^{(1,1)}(x)(1+x)dx = \frac{4}{(m+1)(m+2)} J_m^{(1,1)}(1),$$

and

$$\int_{-1}^{1} J_m^{(1,1)}(x)(1-x)dx = \frac{4}{(m+1)(m+2)} J_m^{(1,1)}(-1).$$

Then

$$\int_{-1}^{1} T_{n+1}^{(1,1)}(x)(1+x)dx = \int_{-1}^{1} \left(J_{n+1}^{(1,1)}(x) - C\,J_n^{(1,1)}(x)\right)(1+x)dx$$

$$= \frac{4}{(n+2)(n+3)} J_{n+1}^{(1,1)}(1) - \frac{4C}{(n+1)(n+2)} J_n^{(1,1)}(1)$$

$$= \frac{4J_n^{(1,1)}(1)}{(n+1)(n+2)} \left(\frac{n+1}{n+3} Q_n^{(1,1)}(1) - C\right) = \frac{4J_n^{(1,1)}(1)}{(n+1)(n+2)} \left(\frac{n+1}{2n+3} - C\right)$$

and, similarly,

$$\int_{-1}^{1} T_{n+1}^{(1,1)}(x)(1-x)dx = \frac{4J_n^{(1,1)}(-1)}{(n+1)(n+2)} \left(-\frac{n+1}{2n+3} - C\right).$$

Hence

$$\mu_{n+1,0} = \frac{4}{(n+1)(n+2)\big(Q_n^{(1,1)}(-1) - C\big)} \left(-\frac{n+1}{2n+3} - C \right)$$

and

$$\mu_{n+1,n+2} = \frac{4}{(n+1)(n+2)\big(Q_n^{(1,1)}(1) - C\big)} \left(\frac{n+1}{2n+3} - C \right).$$

Taking into account (34) we obtain that $\mu_{n+1,0} > 0$ and $\mu_{n+1,n+2} > 0$ if and only if (33) holds.

($\Leftarrow$) If (33) holds, then, in particular, (34) holds and then, from Proposition 1, the polynomial $T_{n+1}^{(1,1)}$ generates a positive $(2n, n+1, (1-x^2)dx)$ quadrature formula for $n > 1$. Hence, for every polynomial $P \in \mathbb{P}_{2n}$

$$\int_{-1}^{1} P(x)(1-x^2)dx = \sum_{k=1}^{n+1} \lambda_{n+1,k} P(y_{n+1,k}),$$

where the points $\{y_{n+1,j}\}_{j=1}^{n+1}$ are the zeros of $T_{n+1}^{(1,1)}$ and $\lambda_{n,k} > 0$ are given by

$$\lambda_{n+1,k} = \int_{-1}^{1} \ell_k(x)(1-x^2)\, dx, \quad k = 1, \ldots, n+1,$$

where $\ell_k(x)$ are the cardinal Lagrange polynomials at the nodes $\{y_{n+1,j}\}_{j=1}^{n+1}$.

Moreover, for each $S \in \mathbb{P}_{n-1}$,

$$\int_{-1}^{1} S(x) T_{n+1}^{(1,1)}(x)(1-x^2)dx = 0\,. \tag{35}$$

Note that, every polynomial $P \in \mathbb{P}_{n+2}$ can be written as

$$P(x) = P(-1)\frac{(1-x)T_{n+1}^{(1,1)}(x)}{2\,T_{n+1}^{(1,1)}(-1)} + (1-x)^2 \sum_{k=1}^{n+1} \frac{P(y_{n+1,k})\ell_k(x)}{1-y_{n+1,k}^2}$$
$$+ P(1)\frac{(1+x)T_{n+1}^{(1,1)}(x)}{2\,T_{n+1}^{(1,1)}(1)}.$$

Hence, for each $P \in \mathbb{P}_{n+2}$,

$$\int_{-1}^{1} P(x)dx = \mu_{n+1,0}P(-1) + \sum_{k=1}^{n+1} \mu_{n+1,k} P(y_{n+1,k}) + \mu_{n+1,n+2}P(1),$$

where, for $1 \le k \le n+1$,

$$\mu_{n+1,k} = \frac{\lambda_{n+1,k}}{1-y_{n+1,k}^2} > 0,$$

$$\mu_{n+1,n+2} = \frac{1}{2T_{n+1}^{(1,1)}(1)} \int_{-1}^{1} T_{n+1}^{(1,1)}(x)(1+x)dx > 0$$

and

$$\mu_{n+1,0} = \frac{1}{2T_{n+1}^{(1,1)}(-1)} \int_{-1}^{1} T_{n+1}^{(1,1)}(x)(1-x)dx > 0,$$

where for the last inequalities we have used the computations given in the first part of the proof.

Finally, if $Q_{2n+2} \in \mathbb{P}_{2n+2}$, then there exist polynomials $S_{n-1} \in \mathbb{P}_{n-1}$ and $P_{n+2} \in \mathbb{P}_{n+2}$ such that $Q_{2n+2}(x) = S_{n-1}(x)(1-x^2)T_{n+1}^{(1,1)}(x) + P_{n+2}(x)$. Taking into account (35) we obtain

$$\begin{aligned}
\int_{-1}^{1} Q_{2n+2}(x)dx &= \int_{-1}^{1} P_{n+2}(x)dx \\
&= \mu_{n+1,0}P_{n+2}(-1) + \sum_{k=1}^{n+1} \mu_{n+1,k}P_{n+2}(y_{n+1,k}) + \mu_{n+1,n+2}P_{n+2}(1) \\
&= \mu_{n+1,0}Q_{2n+2}(-1) + \sum_{k=1}^{n+1} \mu_{n+1,k}\, Q_{2n+2}(y_{n+1,k}) + \mu_{n+1,n+2}Q_{2n+1}(1).
\end{aligned}$$

□

4 Positive Lobatto and Radau quadratures containing a give node

For $\alpha, \beta > -1$, let $\{x_{n,k}^{(\alpha,\beta)}\}_{k=1}^{n}$ be the zeros of $P_n^{(\alpha,\beta)}(x)$. Put $x_{n,0}^{(\alpha,\beta)} = -\infty$ and $x_{n,n+1}^{(\alpha,\beta)} = +\infty$. It is know that the function $Q_n^{(\alpha,\beta)}(x)$ defined in (6) increases from $-\infty$ to $+\infty$ in each one of the intervals $\left(x_{n,k-1}^{(\alpha,\beta)}, x_{n,k}^{(\alpha,\beta)}\right)$, $1 \le k \le n+1$, (see Theorem 3.3.5 of [9], p. 46), and $Q_n^{(\alpha,\beta)}$ is negative on $\left(x_{n,k-1}^{(\alpha,\beta)}, x_{n+1,k}^{(\alpha,\beta)}\right)$ and positive on $\left(x_{n+1,k}^{(\alpha,\beta)}, x_{n,k}^{(\alpha,\beta)}\right)$.

Taking into account the above properties, we define some numbers as follows.

For each k, $1 \le k \le n+1$, let $v_{n+1,k}^{(\alpha,\beta)} \in \left(x_{n,k-1}^{(\alpha,\beta)}, x_{n+1,k}^{(\alpha,\beta)}\right)$ be the only point where

$$Q_n^{(\alpha,\beta)}\left(v_{n+1,k}^{(\alpha,\beta)}\right) = Q_n^{(\alpha,\beta)}(-1) \tag{36}$$

and let $w_{n+1,k}^{(\alpha,\beta)} \in \left(x_{n+1,k}^{(\alpha,\beta)}, x_{n,k}^{(\alpha,\beta)}\right)$ be the only point where

$$Q_n^{(\alpha,\beta)}\left(w_{n+1,k}^{(\alpha,\beta)}\right) = Q_n^{(\alpha,\beta)}(1). \tag{37}$$

Note that $v_{n+1,1}^{(\alpha,\beta)} = -1$ and $w_{n+1,n+1}^{(\alpha,\beta)} = 1$.

We define the intervals $I_{n+1,k}^{(\alpha,\beta)} = \left(v_{n+1,k}^{(\alpha,\beta)}, w_{n+1,k}^{(\alpha,\beta)}\right)$, $k = 1, \ldots, n+1$. Since

$$x_{n,k-1}^{(\alpha,\beta)} < v_{n+1,k}^{(\alpha,\beta)} < x_{n+1,k}^{(\alpha,\beta)} < w_{n+1,k}^{(\alpha,\beta)} < x_{n,k}^{(\alpha,\beta)}, \quad k = 0, \ldots, n,$$

the function $R_{n+1,\theta}^{(\alpha,\beta)}(x)$ (given by (7)) is well defined for $\theta \in \bigcup_{k=1}^{n+1} I_{n+1,k}^{(\alpha,\beta)}$. Moreover, since $Q_n^{(\alpha,\beta)}$ is increasing on $(x_{n,k-1}^{(\alpha,\beta)}, x_{n,k}^{(\alpha,\beta)})$, we obtain

$$Q_n^{(\alpha,\beta)}(-1) = Q_n^{(\alpha,\beta)}(v_{n+1,k}^{(\alpha,\beta)}) < Q_n^{(\alpha,\beta)}(\theta) < Q_n^{(\alpha,\beta)}(w_{n+1,k}^{(\alpha,\beta)}) = Q_n^{(\alpha,\beta)}(1).$$

Hence, we can rewrite Proposition 1 in terms of the sets $I_{n+1,k}^{(\alpha,\beta)}$.

Proposition 2. *Fix $\alpha, \beta > -1$, $n \in \mathbb{N}$, $\theta \in (-1,1)$ and $1 \leq k < n$. The polynomial $R_{n+1,\theta}^{(\alpha,\beta)}(x)$ defined by (7) generates a positive $(2n, n+1, (1-x)^\alpha(1+x)^\beta dx)$ quadrature formula which contains θ as a node if and only if $\theta \in \bigcup_{k=0}^{n} I_{n+1,k}^{(\alpha,\beta)}$.*

Notice that, for a fixed $\alpha, \beta \in \{0,1\}$, we can not cover all the interval $(-1,1)$ with the union of the intervals considered in Proposition 2. Hence, if we want to find a positive quadrature formula for every θ, the parameters α and β must change.

We will prove that the union of the sets $I_{n,k}^{(\alpha,\beta)}$ with $\alpha, \beta \in \{0,1\}$ covers the interval $(-1,1)$. When θ belongs to one of these intervals, we will find a positive quadrature formula associated to a linear combination of some Jacobi polynomials, which contains θ as a node (see Proposition 2). From Theorems 2 and 3 we know that the interval where we can have a Lobatto or a Radau positive quadrature formula is a proper subset of an interval where we have a quadrature associated with a Jacobi weight. Thus we should verify that some of these intervals have non empty intersection.

Taking into account Theorems 2 and 3, for $\alpha, \beta \in \{0,1\}$ and $1 \leq k \leq n+1$, we also consider the points defined by the equations

$$Q_n^{(\alpha,\beta)}\big(y_{n+1,k}^{(\alpha,\beta)}\big) = -\frac{n+1}{2n+3}, \quad y_{n+1,k}^{(\alpha,\beta)} \in \left(v_{n+1,k}^{(\alpha,\beta)}, x_{n+1,k}^{(\alpha,\beta)}\right),$$

$$Q_n^{(\alpha,\beta)}\big(z_{n+1,k}^{(\alpha,\beta)}\big) = \frac{n+1}{2n+3}, \quad z_{n+1,k}^{(\alpha,\beta)} \in \left(x_{n+1,k}^{(\alpha,\beta)}, w_{n+1,k}^{(\alpha,\beta)}\right).$$

From a Theorem of A. Markov (see Theorem 3.3.4 of [9], p. 111) we know that the function

$$F_{n,k}(\alpha,\beta) = x_{n,k}^{(\alpha,\beta)}, \qquad 1 \leq k \leq n, \;\; \alpha > -1, \;\; \beta > -1,$$

decreases with α and increases with β. Moreover, from Theorem 2.3 in [3] we have, for $1 \leq k \leq n$,

$$x_{n+1,k}^{(\alpha,\beta)} < x_{n,k}^{(\alpha+s,\beta+t)} < x_{n+1,k+1}^{(\alpha,\beta)}, \quad \text{for all } \alpha, \beta > -1 \text{ and } 0 \leq s, t \leq 2. \tag{38}$$

We do not know if the interlacing property given in (39) is a new result.

Proposition 3. *If $n \in \mathbb{N}$, then*

i)

$$v_{n+1,k+1}^{(1,0)} = x_{n,k}^{(1,1)} = w_{n+1,k}^{(0,1)}, \qquad 1 \leq k \leq n,$$

and

$$w_{n+1,k}^{(0,0)} = x_{n,k}^{(1,0)} \qquad \textit{and} \qquad v_{n+1,k+1}^{(0,0)} = x_{n,k}^{(0,1)}, \qquad 1 \le k \le n.$$

In particular,

$$x_{n,1}^{(1,0)} < x_{n,1}^{(0,1)} < x_{n,2}^{(1,0)} < x_{n,2}^{(0,1)} < \ldots < x_{n,n}^{(1,0)} < x_{n,n}^{(0,1)}. \tag{39}$$

ii) For $1 \le k \le n+1$, *one has*

$$x_{n+1,k}^{(1,0)} < x_{n+1,k}^{(0,0)} = z_{n+1,k}^{(1,0)} < w_{n+1,k}^{(1,0)} < x_{n,k}^{(1,0)},$$

$$x_{n,k-1}^{(0,1)} < v_{n+1,k}^{(0,1)} < y_{n+1,k}^{(0,1)} = x_{n+1,k}^{(0,0)} < x_{n+1,k}^{(0,1)},$$

$$x_{n,k-1}^{(1,1)} < v_{n+1,k}^{(1,1)} < y_{n+1,k}^{(1,1)} = x_{n+1,k}^{(1,0)} < x_{n+1,k}^{(1,1)},$$

and

$$x_{n+1,k}^{(1,1)} < x_{n+1,k}^{(0,1)} = z_{n+1,k}^{(1,1)} < w_{n+1,k}^{(1,1)} < x_{n,k}^{(1,1)}.$$

Proof. i) It is known that (see equation (4.5.4) in [9] p. 71)

$$P_n^{(1,1)}(x) = \frac{2(n+1)}{2n+3}\,\frac{P_n^{(0,1)}(x) - P_{n+1}^{(0,1)}(x)}{1-x} \tag{40}$$

and

$$P_n^{(1,1)}(x) = \frac{2(n+1)}{2n+3}\,\frac{P_n^{(1,0)}(x) + P_{n+1}^{(1,0)}(x)}{1+x}. \tag{41}$$

In particular, from (40) and (41) we have

$$P_{n+1}^{(0,1)}(x_{n,k}^{(1,1)}) = P_n^{(0,1)}(x_{n,k}^{(1,1)}) \quad \text{and} \quad P_{n+1}^{(1,0)}(x_{n,k}^{(1,1)}) = -P_n^{(1,0)}(x_{n,k}^{(1,1)}).$$

Therefore, for $1 \le k \le n$, one has (see (9))

$$\begin{aligned} Q_n^{(0,1)}\big(x_{n,k}^{(1,1)}\big) &= \frac{J_{n+1}^{(0,1)}(x_{n,k}^{(1,1)})}{J_n^{(0,1)}(x_{n,k}^{(1,1)})} = \frac{l_n^{(0,1)} P_{n+1}^{(0,1)}(x_{n,k}^{(1,1)})}{l_{n+1}^{(1,0)} P_n^{(0,1)}(x_{n,k}^{(1,1)})} \\ &= \frac{l_n^{(0,1)}}{l_{n+1}^{(0,1)}} = \frac{l_n^{(0,1)} P_{n+1}^{(0,1)}(1)}{l_{n+1}^{(0,1)} P_n^{(0,1)}(1)} = Q_n^{(0,1)}(1). \end{aligned}$$

Since the zeros of $P_n^{(1,1)}$ are symmetric, we also have $Q_n^{(0,1)}(-x_{n,k}^{(1,1)}) = Q_n^{(0,1)}(1)$. Then, from (19), we also obtain

$$Q_n^{(1,0)}(x_{n,k}^{(1,1)}) = Q_n^{(1,0)}(-1).$$

Moreover, it follows from the first inequality in (38) (with $\alpha = 0,\ \beta = 1,\ s = 1$ and $t = 0$) and Markov's theorem that

$$x_{n+1,k}^{(0,1)} < x_{n,k}^{(1,1)} < x_{n,k}^{(0,1)}.$$

Since $Q_n^{(0,1)}$ is increasing in the interval $\left(x_{n+1,k}^{(0,1)}, x_{n,k}^{(0,1)}\right)$, we get

$$w_{n+1,k}^{(0,1)} = x_{n,k}^{(1,1)}, \quad k = 1, \ldots, n.$$

On the other hand, from Markov's theorem and the second inequality in (38) with $\alpha = 1, \beta = 0$, $s = 0$ and $t = 1$, we know that

$$x_{n,k}^{(1,0)} < x_{n,k}^{(1,1)} < x_{n+1,k+1}^{(1,0)}.$$

Since $Q_n^{(1,0)}$ is increasing, we have

$$v_{n+1,k+1}^{(1,0)} = x_{n,k}^{(1,1)}, \quad k = 1, \ldots, n.$$

It follows from equation (4.5.3) in [9] (p. 71, with $\alpha = \beta = 0$) that

$$P_n^{(1,0)}(x) = \frac{P_n^{(0,0)}(x) - P_{n+1}^{(0,0)}(x)}{1-x} \text{ and } P_n^{(0,1)}(x) = \frac{P_n^{(0,0)}(x) + P_{n+1}^{(0,0)}(x)}{1+x}. \quad (42)$$

In particular, from (42) we have

$$P_{n+1}^{(0,0)}(x_{n,k}^{(1,0)}) = P_n^{(0,0)}(x_{n,k}^{(1,0)}) \quad \text{and} \quad P_{n+1}^{(0,0)}(x_{n,k}^{(0,1)}) = -P_n^{(0,0)}(x_{n,k}^{(0,1)}).$$

Therefore,

$$Q_n^{(0,0)}(x_{n,k}^{(1,0)}) = \frac{J_{n+1}^{(0,0)}(x_{n,k}^{(1,0)})}{J_n^{(0,0)}(x_{n,k}^{(1,0)})} = \frac{l_n^{(0,0)} P_{n+1}^{(0,0)}(x_{n,k}^{(1,0)})}{l_{n+1}^{(0,0)} P_n^{(0,0)}(x_{n,k}^{(1,0)})} = \frac{l_n^{(0,0)}}{l_{n+1}^{(0,0)}} = Q_n^{(0,0)}(1).$$

Moreover, from the first inequality in (38) (with $\alpha = \beta = 0$, $s = 1$ and $t = 0$) and Markov's theorem, one has

$$x_{n+1,k}^{(0,0)} < x_{n,k}^{(1,0)} < x_{n,k}^{(0,0)},$$

and then $w_{n+1,k}^{(0,0)} = x_{n,k}^{(1,0)}$, $k = 1, \ldots, n$.

In a similar way, we obtain

$$Q_n^{(0,0)}(x_{n,k}^{(0,1)}) = Q_n^{(0,0)}(-1).$$

From Markov's theorem and the second inequality in (38) (with $\alpha = 0, \beta = 0$, $s = 0$ and $t = 1$) we know that

$$x_{n,k}^{(0,0)} < x_{n,k}^{(0,1)} < x_{n+1,k+1}^{(0,0)},$$

and then $v_{n+1,k+1}^{(0,0)} = x_{n,k}^{(0,1)}$, $k = 1, \ldots, n$.

Finally, since

$$x_{n+1,k}^{(0,0)} < w_{n+1,k}^{(0,0)} = x_{n,k}^{(1,0)} < x_{n,k}^{(0,0)} < v_{n+1,k+1}^{(0,0)} = x_{n,k}^{(0,1)} < x_{n+1,k+1}^{(0,0)},$$

we have proved (39).

ii) From the recurrence relation (15) we obtain

$$\frac{P_{n+2}^{(0,0)}(x_{n+1,k}^{(0,0)})}{P_n^{(0,0)}(x_{n+1,k}^{(0,0)})} = -\frac{n+1}{n+2}.$$

Thus (11) and (42) yields

$$\begin{aligned} Q_n^{(1,0)}(x_{n+1,k}^{(0,0)}) &= \frac{J_{n+1}^{(1,0)}(x_{n+1,k}^{(0,0)})}{J_n^{(1,0)}(x_{n+1,k}^{(0,0)})} = \frac{l_n^{(1,0)}}{l_{n+1}^{(1,0)}} \frac{P_{n+1}^{(1,0)}(x_{n+1,k}^{(0,0)})}{P_n^{(1,0)}(x_{n+1,k}^{(0,0)})} \\ &= \frac{n+2}{2n+3} \frac{P_{n+1}^{(0,0)}(x_{n+1,k}^{(0,0)}) - P_{n+2}^{(0,0)}(x_{n+1,k}^{(0,0)})}{P_n^{(0,0)}(x_{n+1,k}^{(0,0)}) - P_{n+1}^{(0,0)}(x_{n+1,k}^{(0,0)})} \\ &= \frac{n+2}{2n+3} \frac{-P_{n+2}^{(0,0)}(x_{n+1,k}^{(0,0)})}{P_n^{(0,0)}(x_{n+1,k}^{(0,0)})} = \frac{n+1}{2n+3} = Q_n^{(1,0)}(z_{n,k}^{(1,0)}). \qquad (43) \end{aligned}$$

From Markov's theorem and the first inequality in (38) (with $\alpha = \beta = 0$, $s = 1$ and $t = 0$) we know that

$$x_{n+1,k}^{(1,0)} < x_{n+1,k}^{(0,0)} < x_{n,k}^{(1,0)}.$$

Since $Q_n^{(1,0)}$ increases in $(x_{n+1,k}^{(1,0)}, x_{n,k}^{(1,0)})$, one has

$$z_{n+1,k}^{(1,0)} = x_{n,k}^{(0,0)} < w_{n+1,k}^{(1,0)}, \quad k = 1, \dots, n+1.$$

Since the zeros of $P_{n+1}^{(0,0)}$ are symmetric and the numbers in (43) does not depend on k, we also have

$$Q_n^{(1,0)}(-x_{n+1,k}^{(0,0)}) = \frac{n+1}{2n+3}.$$

Then, from (19) we obtain

$$\begin{aligned} Q_n^{(0,1)}(x_{n+1,k}^{(0,0)}) = -\frac{n+1}{2n+3} = Q_n^{(0,1)}(y_{n+1,k}^{(0,1)}) > \\ -\frac{(n+2)^2}{(n+1)(2n+3)} = Q_n^{(0,1)}(-1) = Q_n^{(0,1)}(v_{n+1,k}^{(0,1)}). \end{aligned}$$

From the second inequality in (38) (with $\alpha = \beta = 0$, $s = 0$ and $t = 1$) and Markov's theorem we have

$$x_{n,k-1}^{(0,1)} < x_{n+1,k}^{(0,0)} < x_{n+1,k}^{(0,1)}.$$

Hence, $y_{n+1,k}^{(0,1)} = x_{n+1,k}^{(0,0)} \in (x_{n-1,k}^{(0,1)}, x_{n+1,k}^{(0,0)})$, $k = 1, \dots, n+1$.

From (41) and the recurrence relation (15) we know that

$$\frac{P_{n+1}^{(1,1)}(x_{n+1,k}^{(1,0)})}{P_{n}^{(1,1)}(x_{n+1,k}^{(1,0)})} = \frac{(n+2)(2n+3)}{(n+1)(2n+5)} \frac{P_{n+1}^{(1,0)}(x_{n+1,k}^{(1,0)}) + P_{n+2}^{(1,0)}(x_{n+1,k}^{(1,0)})}{P_{n}^{(1,0)}(x_{n+1,k}^{(1,0)}) + P_{n+1}^{(1,0)}(x_{n+1,k}^{(1,0)})}$$

$$= \frac{(n+2)(2n+3)}{(n+1)(2n+5)} \frac{P_{n+2}^{(1,0)}(x_{n+1,k}^{(1,0)}))}{P_{n}^{(1,0)}(x_{n+1,k}^{(1,0)}))}$$

$$= -\frac{(n+2)(2n+3)}{(n+1)(2n+5)} \frac{(n+1)(2n+5)}{(n+3)(2n+3)} = -\frac{n+2}{n+3}.$$

Therefore

$$Q_{n+1}^{(1,1)}(x_{n+1,k}^{(1,0)}) = \frac{l_n^{(1,1)}}{l_{n+1}^{(1,1)}} \frac{P_{n+1}^{(1,1)}(x_{n+1,k}^{(1,0)})}{P_{n}^{(1,1)}(x_{n+1,k}^{(1,0)})}$$

$$= -\frac{(n+1)(n+3)}{(n+2)(2n+3)} \frac{n+2}{n+3}$$

$$= -\frac{n+1}{2n+3} = Q_n^{(1,1)}(y_{n+1,k}^{(1,1)}) > -\frac{n+3}{2n+3} = Q_n^{(1,1)}(-1) = Q_n^{(1,1)}(v_{n+1,k}^{(1,1)}).$$

From the second inequality in (38) (with $\alpha = 1$, $\beta = 0$, $s = 0$ and $t = 1$) and Markov's theorem we know that

$$x_{n,k-1}^{(1,1)} < x_{n+1,k}^{(1,0)} < x_{n+1,k}^{(1,1)}.$$

Hence, $y_{n+1,k}^{(1,1)} = x_{n+1,k}^{(1,0)}$, $k = 1, \ldots, n+1$.

Finally, from (40) and the recurrence relation (15) we obtain

$$\begin{aligned}
\frac{P_{n+1}^{(1,1)}(x_{n+1,k}^{(0,1)})}{P_{n}^{(1,1)}(x_{n+1,k}^{(0,1)})} &= \frac{(n+2)(2n+3)}{(n+1)(2n+5)} \frac{P_{n+1}^{(0,1)}(x_{n+1,k}^{(0,1)}) - P_{n+2}^{(0,1)}(x_{n+1,k}^{(0,1)})}{P_{n}^{(0,1)}(x_{n+1,k}^{(0,1)}) - P_{n+1}^{(0,1)}(x_{n+1,k}^{(0,1)})} \\
&= -\frac{(n+2)(2n+3)}{(n+1)(2n+5)} \frac{P_{n+2}^{(0,1)}(x_{n+1,k}^{(0,1)}))}{P_{n}^{(0,1)}(x_{n+1,k}^{(0,1)}))} \\
&= \frac{(n+2)(2n+3)}{(n+1)(2n+5)} \frac{(n+1)(2n+5)}{(n+3)(2n+3)} = \frac{n+2}{n+3}.
\end{aligned}$$

Therefore (see (11))

$$Q_{n+1}^{(1,1)}(x_{n+1,k}^{(0,1)}) = \frac{l_n^{(1,1)}}{l_{n+1}^{(1,1)}} \frac{P_{n+1}^{(1,1)}(x_{n+1,k}^{(0,1)})}{P_{n}^{(1,1)}(x_{n+1,k}^{(0,1)})} = \frac{(n+1)(n+3)}{(n+2)(2n+3)} \frac{n+2}{n+3}$$

$$= \frac{n+1}{2n+3} = Q_n^{(1,1)}(z_{n+1,k}^{(1,1)}) < \frac{n+3}{2n+3} = Q_{n+1}^{(1,1)}(1) = Q_n^{(1,1)}(w_{n+1,k}^{(1,1)}).$$

From Markov's theorem and the first inequality in (38) (with $\alpha = 0$, $\beta = 1$, $s = 1$ and $t = 0$) we know that

$$x_{n+1,k}^{(1,1)} < x_{n+1,k}^{(0,1)} < x_{n,k}^{(1,1)}.$$

Hence, $z_{n+1,k}^{(1,1)} = x_{n+1,k}^{(0,1)}$, $k = 1, \ldots, n+1$. ∎

5 Main results

Theorem 5. *Fix $\theta \in (-1,1)$ and $n \in \mathbb{N}$. The polynomial $R^{(1,1)}_{n+1,\theta}(x)$ generates a Lobatto positive $(2n+2, n+1, dx)$ quadrature formula as in (4) if and only if*

$$\theta \in \bigcup_{k=1}^{n+1} \left(x^{(1,0)}_{n+1,k}, x^{(0,1)}_{n+1,k} \right).$$

Proof. From Theorem 4, we know that the polynomial $R^{(1,1)}_{n+1,\theta}(x)$ generates a Lobatto positive $(2n+2, n+1, dx)$ quadrature formula if and only if

$$Q^{(1,1)}_n(y^{(1,1)}_{n+1,k}) = -\frac{n+1}{2n+3} < Q^{(1,1)}_n(\theta) < \frac{n+1}{2n+3} = Q^{(1,1)}_n(z^{(1,1)}_{n+1,k}).$$

Since $Q^{(1,1)}_n$ increases, it happen if and only if there exists k $(1 \le k \le n+1)$ such that $\theta \in (y^{(1,1)}_{n+1,k}, z^{(1,1)}_{n+1,k})$. But, $y^{(1,1)}_{n+1,k} = x^{(1,0)}_{n+1,k}$ and $z^{(1,1)}_{n+1,k} = x^{(0,1)}_{n+1,k}$. □

Theorem 6. *Fix $\theta \in (-1,1)$ and $n \in \mathbb{N}$. The polynomial $R^{(1,0)}_{n+1,\theta}(x)$ generates a Radau positive $(2n+1, n+1, dx)$ quadrature formula as in (2) if and only if*

$$\theta \in \left(-1, x^{(0,0)}_{n+1,1}\right) \bigcup \left(\bigcup_{k=1}^{n} \left(x^{(1,1)}_{n,k}, x^{(0,0)}_{n+1,k+1} \right) \right).$$

Proof. From Theorem 2, we know that the polynomial $R^{(1,0)}_{n,\theta}(x)$ generates a Lobatto positive $(2n+1, n+1, dx)$ quadrature formula if and only if

$$Q^{(1,0)}_n(v^{(1,0)}_{n+1,k}) = Q^{(1,0)}_n(-1) = -\frac{n+2}{2n+3} < Q^{(1,0)}_n(\theta) < \frac{n+1}{2n+3} = Q^{(1,0)}_n(z^{(1,0)}_{n+1,k}).$$

Since $Q^{(1,0)}_n$ increases, it happen if and only if there exists k $(0 \le k \le n+1)$ such that $\theta \in (v^{(1,0)}_{n+1,k}, z^{(1,0)}_{n+1,k})$. That is,

$$\theta \in \bigcup_{k=1}^{n+1} (v^{(1,0)}_{n+1,k}, z^{(1,0)}_{n+1,k}) = (v^{(1,0)}_{n+1,1}, z^{(1,0)}_{n+1,1}) \bigcup \left(\bigcup_{k=1}^{n} (v^{(1,0)}_{n+1,k+1}, z^{(1,0)}_{n+1,k+1}) \right).$$

But, $v^{(1,0)}_{n+1,1} = -1$, $v^{(1,1)}_{n+1,k+1} = x^{(1,1)}_{n,k}$ and $z^{(1,0)}_{n+1,k} = x^{(0,0)}_{n+1,k}$.

□

Theorem 7. *Fix $\theta \in (-1,1)$ and $n \in \mathbb{N}$. The polynomial $R^{(0,1)}_{n+1,\theta}(x)$ generates a Radau positive $(2n+1, n+1, dx)$ quadrature formula as in (3) if and only if*

$$\theta \in \left(\bigcup_{k=1}^{n} \left(x^{(0,0)}_{n+1,k}, x^{(1,1)}_{n,k} \right) \right) \bigcup \left(x^{(0,0)}_{n+1,n+1}, 1 \right).$$

Proof. From Theorem 2, we know that the polynomial $R^{(0,1)}_{n+1,\theta}(x)$ generates a Lobatto positive $(2n+1, n+1, dx)$ quadrature formula if and only if

$$Q_n^{(0,1)}(y^{(0,1)}_{n+1,k}) = -\frac{n+1}{2n+3} < Q_n^{(0,1)}(\theta) < \frac{n+2}{2n+3} = Q_n^{(0,1)}(1) = Q_n^{(0,1)}(w^{(0,1)}_{n+1,k}).$$

Since $Q_n^{(0,1)}$ increases, it happen if and only if there exists k $(1 \le k \le n+1)$ such that $\theta \in (y^{(0,1)}_{n+1,k}, w^{(0,1)}_{n+1,k})$. That is,

$$\theta \in \bigcup_{k=1}^{n+1} (y^{(0,1)}_{n+1,k}, w^{(0,1)}_{n+1,k}) = \left(\bigcup_{k=1}^{n} (y^{(0,1)}_{n+1,k}, w^{(0,1)}_{n+1,k})\right) \bigcup (y^{(0,1)}_{n+1,n+1}, w^{(0,1)}_{n+1,n+1}).$$

But, $w^{(0,1)}_{n+1,n+1} = 1$, $w^{(0,1)}_{n+1,k} = x^{(1,1)}_{n,k}$, $k = 1, \dots n$, and $y^{(0,1)}_{n+1,k} = x^{(0,0)}_{n+1,k}$, $k = 1, \dots, n+1$.

□

Theorem 8. *Fix $\theta \in (-1,1)$ and $n \in \mathbb{N}$. The polynomial $R^{(0,0)}_{n+1,\theta}(x)$ generates a positive $(2n, n+1, dx)$ quadrature formula as in* (1) *if and only if*

$$\theta \in \left(-1, x^{(1,0)}_{n,1}\right) \bigcup \left(\bigcup_{k=2}^{n-1} \left(x^{(0,1)}_{n,k}, x^{(1,0)}_{n,k+1}\right)\right) \bigcup \left(x^{(0,1)}_{n,n}, 1\right).$$

Proof. From Proposition 1, we know that the polynomial $R^{(0,0)}_{n+1,\theta}(x)$ generates a positive $(2n, n+1, dx)$ quadrature formula if and only if

$$Q_n^{(0,0)}(v^{(0,0)}_{n+1,k}) = Q_n^{(0,0)}(-1) < Q_n^{(0,0)}(\theta) < Q_n^{(0,0)}(1) = Q_n^{(0,1)}(w^{(0,0)}_{n+1,k}).$$

Since $Q_n^{(0,0)}$ increases, the conditions given above holds if and only if there exists k $(1 \le k \le n+1)$ such that $\theta \in (v^{(0,0)}_{n+1,k}, w^{(0,0)}_{n+1,k})$. That is,

$$\theta \in \left(v^{(0,0)}_{n+1,1}, w^{(0,0)}_{n+1,1}\right) \bigcup \left(\bigcup_{k=1}^{n-1} (v^{(0,0)}_{n+1,k+1}, w^{(0,0)}_{n+1,k+1})\right) \bigcup \left(v^{(0,0)}_{n+1,n+1}, w^{(0,0)}_{n+1,n+1}\right).$$

But, $v^{(0,0)}_{n+1,1} = -1$, $v^{(0,0)}_{n+1,k+1} = x^{(0,1)}_{n,k}$ $(1 \le k \le n)$, $w^{(0,0)}_{n+1,k} = x^{(1,0)}_{n,k}$ $(1 \le k < n)$ and $w^{(0,0)}_{n+1,n+1} = 1$. □

Remark 1. A direct computation shows that Theorems 5, 6 and 7 also holds for $n = 0$.

Corollary 1. *For all $m \in \mathbb{N}$ and for all $\theta \in (-1,1)$ there exist $k \in \mathbb{N}_0$ and $\alpha, \beta \in \{0,1\}$* (*uniquely determined*) *such that $R^{(\alpha,\beta)}_{k,\theta}$ generates a quadrature formula of order of exactness m.*

Proof. Assume $m = 2n + 1$ with $n \in \mathbb{N}_0$. If $P_{n+1}^{(0,0)}(\theta) = 0$ then it is know that $P_{n+1}^{(0,0)}$ generates a Gauss quadrature formula of order of exactness $2n+1$. Since $P_n^{(0,0)}(\theta) \neq 0$, we have

$$R_{n+1,\theta}^{(0,0)}(x) = C_{n,\theta} P_{n+1}^{(0,0)}(x),$$

for some $C_{n,\theta} \neq 0$, and the result holds by taking $\alpha = \beta = 0$ and $k = n + 1$.

A similar result follows if θ is a zero of $P_{n,\theta}^{(1,1)}$.

Note that the intervals in Theorems 6 and 7 are disjoints and cover the interval $(-1, 1)$ unless the points $\{x_{n,k}^{(1,1)}\}_{k=1}^{n}$ and $\{x_{n+1,k}^{(0,0)}\}_{k=1}^{n+1}$. So, setting $x_{n,0}^{(1,1)} = -1$ and $x_{n,n+1}^{(1,1)} = 1$, we have the following situations:

(i) $\theta \in \left(x_{n,k}^{(1,1)}, x_{n+1,k+1}^{(0,0)}\right)$ for some $k = 0, 1, \cdots, n$,

(ii) $\theta \in \left(x_{n+1,k}^{(0,0)}, x_{n,k}^{(1,1)}\right)$ for some $k = 1, \cdots, n + 1$,

(iii) $\theta = x_{n,k}^{(1,1)}$ for some $k = 1, \ldots, n$, or

(iv) $\theta = x_{n+1,k}^{(0,0)}$ for some $k = 1, \ldots, n + 1$.

In the case (i) Theorem 6 assert us that the polynomial $R_{n+1,\theta}^{(1,0)}$ provides a Radau quadrature formula of order of exactness $2n + 1$.

The case (ii) follows analogously from Theorem 7 by using the polynomial $R_{n+1,\theta}^{(0,1)}$. We also have a Radau quadrature formula of order of exactness $2n+1$. In the case (iii), Theorem 3 provides a Lobato quadrature of order of exactness $2n + 1$ that contains the point θ as a node. This formula has n interior nodes and two extremal nodes.

Finally, in the case (iv) it is know that the zeros of the polynomial $P_{n+1}^{((0,0)}$ provide a Gauss quadrature formula of order of exactness $2n + 1$.

The proof follows analogously when $m = 2n$, $n \geq 1$, by using Theorem 5 (with n replaced by $n - 1$) and Theorem 8, when θ is not a zero of $P_n^{(0,1)}$ nor $P_n^{(1,0)}$, and from Theorem 1 otherwise. ☐

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